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EXISTENCE OF SOLUTIONS FOR ELLIPTIC PROBLEMS WITH CRITICAL SOBOLEV–HARDY EXPONENTS

BY

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ABSTRACT

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain such that $0 \in \Omega$, $N \geq 3$, $0 \leq s < 2$, $2^*(s) = 2(N-s)/(N-2)$. We prove the existence of nontrival solutions for the singular critical problem

$$-\Delta u - \mu \frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}}{|x|^s} u + \lambda u$$

with Dirichlet boundary condition on Ω for suitable positive parameters λ and μ.

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1. Introduction and main results

In recent years, people have paid much attention to the existence of nontrivial solutions for the singular problem

$$(1.1) \qquad \begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = |u|^{2^*-2}u + \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 3), 0 \in \Omega, \lambda > 0$, $0 \leq \mu < \bar{\mu} \triangleq ((N-2)/2)^2, 2^* \triangleq 2N/(N-2)$ is the critical Sobolev exponent. As a consequence of Hardy's inequality (see [8]), the linear elliptic operator $L \triangleq (-\Delta - \mu/|x|^2)$ is positive and has discrete spectrum σ_μ in $H_0^1(\Omega)$ if $0 \leq \mu < \bar{\mu}$. Let λ_1 be the first eigenvalue of the operator L in $H_0^1(\Omega)$ and set

$$(1.2) \qquad J_{2^*}(u) \triangleq \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2} - \lambda u^2\right) - \frac{1}{2^*}\int_\Omega |u|^{2^*}, \quad \forall u \in H_0^1(\Omega).$$

Due to the invariance of H_0^1-norm, L^{2^*}-norm and $\int_\Omega u^2/|x|^2$ with respect to rescaling $u \mapsto u_\varepsilon = \varepsilon^{(N-2)/2}u(\varepsilon(\cdot))$ and the existence of the non-trivial entire solution of the limiting problem (see [3], [6], [7] and [10])

$$(1.3) \qquad \begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = |u|^{2^*-2}u, & x \in \mathbb{R}^N, \\ u \to 0, & |x| \to \infty, \end{cases}$$

J_{2^*} fails to satisfy the classical Palais–Smale (PS in short) condition in $H_0^1(\Omega)$. However, a local PS condition can be established. Indeed, let $|u|_p^p = \int_\Omega |u|^p$ for $p \in (1, \infty)$ and

$$(1.4) \qquad A \triangleq \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla u|^2 - \mu\frac{u^2}{|x|^2})}{(\int_\Omega |u|^{2^*})^{2/2^*}}.$$

Suppose $\{u_n\} \subset H_0^1(\Omega)$ is a sequence such that $J_{2^*}(u_n) \leq c < \frac{1}{N}A^{N/2}, J'_{2^*}(u_n) \to 0$ in $H^{-1}(\Omega) = (H_0^1(\Omega))^*$. Then $\{u_n\}$ contains a strongly convergent subsequence; see also [12], [13], [14] and [15]. Using this local PS condition, Jannelli proved in [10] that problem (1.1) has at least one positive solution $u_0 \in H_0^1(\Omega)$ if either (1) $\mu \in (0, \bar{\mu} - 1)$ and $\lambda \in (0, \lambda_1)$ or (2) $\mu \in (\bar{\mu} - 1, \bar{\mu})$ and $\lambda \in (\lambda_*, \lambda_1)$ holds, where λ_* is a positive constant depending on μ. Also, by the compactness analysis argument, Ferrero and Gazzola in [7] investigated the existence of nontrival solutions to (1.1) for a large range of λ; Ghoussoub and Yuan in [9] and Ekeland and Ghoussoub in [6] studied a more general case. Recently, Cao and Peng in [2] proved the existence of sign-changing solutions for problem (1.1) by applying the min-max principles. Catrina and Wang in [3] and Terracini in [18] proved that for $\beta \triangleq \sqrt{\bar{\mu}-\mu}, \varepsilon > 0$ and a suitable $C > 0$, the functions

$$Y_\varepsilon = \frac{C\varepsilon^{(N-2)/2}}{|x|^{\sqrt{\bar{\mu}}-\beta}(\varepsilon^2 + |x|^{4\beta/(N-2)})^{\sqrt{\bar{\mu}}}}$$

satisfy equation (1.3); moreover, Y_ε achieve A on $\mathbb{R}^N$.

Now we consider the following problem,

$$\begin{cases} -\Delta u - \mu\frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}}{|x|^s}u + \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, $0 \in \Omega$, $0 \leq \mu < \bar{\mu}$, $\lambda > 0$, $0 \leq s < 2$, $2^*(s) \triangleq 2(N-s)/(N-2)$ is the critical Sobolev–Hardy exponent; note that $2^*(0) = 2^*$ is the critical Sobolev exponent and as $s = 0$, (1.5) becomes (1.1).

Thus problem (1.5) is in fact the continuation of problem (1.1). In the case of problem (1.5), we need to consider not only the effect of parameter λ and μ, but also that of parameter s. Problem (1.5) is more complicated to deal with than problem (1.1).

A natural interesting question is whether the results about the solutions of (1.1) remain true for (1.5) as $0 < s < 2$, with the critical Sobolev–Hardy growth?

Recently, Kang and Peng in [11] proved that (1.5) has positive solutions and sign-changing solutions for suitable $\mu \in [0, \bar{\mu})$, $\lambda \in (0, \lambda_1)$ and $s \in [0, 2)$. Moreover, they found that for $\varepsilon > 0$ and $\beta = \sqrt{\bar{\mu}-\mu}$, the functions

$$u_\varepsilon^*(x) = \Big(\frac{2\varepsilon^2\beta^2(N-s)}{\sqrt{\bar{\mu}}}\Big)^{\sqrt{\bar{\mu}}/(2-s)} \Big/ \Big(|x|^{\sqrt{\bar{\mu}}-\beta}\Big(\varepsilon^2 + |x|^{(2-s)\beta/\sqrt{\bar{\mu}}}\Big)^{(N-2)/(2-s)}\Big) \tag{1.6}$$

solve the equation

$$-\Delta u - \mu\frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}}{|x|^s}u \quad \text{in } \mathbb{R}^N \setminus \{0\}$$

and satisfy

$$\int_{\mathbb{R}^N}\Big(|\nabla u_\varepsilon^*|^2 - \mu\frac{|u_\varepsilon^*|^2}{|x|^2}\Big) = \int_{\mathbb{R}^N}\frac{|u_\varepsilon^*|^{2^*(s)}}{|x|^s} = A_s^{(N-s)/(2-s)}, \tag{1.7}$$

where A_s is the best constant defined as

$$A_s \triangleq \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega(|\nabla u|^2 - \mu\frac{u^2}{|x|^2})}{(\int_\Omega \frac{u^{2^*(s)}}{|x|^s})^{2/2^*(s)}}; \tag{1.8}$$

A_s is independent of Ω and is achieved by u_ε^* only on $\mathbb{R}^N$. Furthermore, $A_0 = A$ is the best constant defined in (1.4).

By Pohozaev's identity (see [8]), if Ω is a star-shaped domain in $\mathbb{R}^N$, then problem (1.5) has no nontrivial solutions for $\lambda \leq 0$. It is easy to verify that as $\lambda \geq \lambda_1$, every solution of (1.5) must change sign. So it will be meaningful to study the existence of nontrival solutions for problem (1.5) as $s \in [0,2)$ and $\lambda \in (0,+\infty)$, especially as $\lambda \in [\lambda_1,+\infty)$. In this paper, we obtain the following existence results.

THEOREM 1.1: *Suppose $N \geq 4, \mu \in [0,\bar{\mu}-1]$, $s \in [0,2)$ and $\lambda \in (\lambda_k,\lambda_{k+1})$ with $\lambda_k,\lambda_{k+1} \in \sigma_\mu$. Then problem (1.5) has at least a pair of sign-changing solutions $\pm u(x)$ in $H_0^1(\Omega)$.*

THEOREM 1.2: *Suppose $N \geq 4, \mu \in (\bar{\mu}-1,\bar{\mu}), s \in [0,2)$ and there exists $\lambda_k \in \sigma_\mu$ such that $\lambda \in (\lambda^*,\lambda_k) \cap (0,+\infty)$, where*

$$\lambda^* = \lambda_k - \left(A_s \int_\Omega |x|^{2s/(2^*(s)-2)}\right)^{-(2-s)/(N-s)}.$$

Then problem (1.5) has ν_k pairs of nontrival solutions, where ν_k denotes the multiplicity of λ_k.

THEOREM 1.3: *Suppose $N \geq 5$, $\Omega = B_1(0)$ is the unit ball, $0 \leq \mu < \bar{\mu} - ((N+2)/N)^2$ and $\lambda = \lambda_1$. Then problem (1.5) has at least a pair of sign-changing solutions $\pm u(x)$ in $H_0^1(\Omega)$ with energy level in the range of $(0, \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)})$.*

It should be mentioned that when $s = 0$, our above results are the same as those in [7]. When $0 < s < 2$, our results are new.

This paper is organized as follows. In Section 2, we establish some asymptotic estimates; in Section 3, we describe the variational procedure; in Section 4, we give the proofs of our theorems. This idea is essentially introduced in [7].

2. Some technical asymptotic estimates

We first define the equivalent norm in $H_0^1(\Omega)$ for $0 \leq \mu < \bar{\mu}$:

$$\|u\| \triangleq \left(\int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\right)\right)^{\frac{1}{2}}, \quad \forall u \in H_0^1(\Omega).$$

By Hardy's inequality, this norm is equivalent to the usual norm in $H_0^1(\Omega)$. We also denote the norm of $L^p(\Omega)$ space as $|u|_p$ and various positive constants as

C. Define

$$J(u) \triangleq \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2} - \lambda u^2\right) - \frac{1}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}, \quad \forall u \in H_0^1(\Omega);$$

then $J \in C^1(H_0^1(\Omega), \mathbb{R})$ and the critical points of functional J correspond to the solutions of (1.5).

Fix $k \in \mathbb{N}$, and for all $i \in \mathbb{N}$ denote by e_i an L^2 normalized eigenfunction relative to $\lambda_i \in \sigma_\mu$; let H^- denote the space spanned by the eigenfunctions corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_k$ and $H^+ \triangleq (H^-)^\perp, P_k\colon H_0^1(\Omega) \mapsto H^-$ denote the orthogonal projection. Take always $m \in \mathbb{N}$ large enough so that $B_{1/m} \subset \Omega$, where $B_{1/m}$ denotes the ball of radius $1/m$ with center at 0. Define

$$\zeta_m(x) \triangleq \begin{cases} 0, & x \in B_{1/m}, \\ m|x| - 1, & x \in A_m = B_{2/m} \setminus B_{1/m}, \\ 1, & x \in \Omega \setminus B_{2/m}, \end{cases}$$

$e_i^m \triangleq \zeta_m e_i, H_m^- \triangleq \operatorname{span}\{e_i^m; i = 1, \ldots, k\}, \gamma \triangleq \sqrt{\bar{\mu}} + \beta$ and $\gamma' \triangleq \sqrt{\bar{\mu}} - \beta$.

Lemma 2.1 ([7]): *As $m \to \infty$, we have*

$$e_i^m \to e_i \quad \text{in } H_0^1(\Omega), \quad \forall i \in \mathbb{N}.$$

Furthermore,

(i) *for $H_m^- = \operatorname{span}\{e_i^m; i = 1, \ldots, k\}$ and $\Lambda = \{u \in H_m^-; |u|_2 = 1\}$, we have*

$$\max_{u \in \Lambda} \|u\|^2 \le \lambda_k + o(1);$$

(ii) *for $H_m^- = \operatorname{span}\{e_1^m\}, \Lambda = \{u \in H_m^-; |u|_2 = 1\}$ and $\Omega = B_1(0)$, we have*

$$\max_{u \in \Lambda} \|u\|^2 \le \lambda_1 + Cm^{-2\beta}.$$

Consider the function $u_\varepsilon^*(x)$ in (1.6); since u_ε^* is a radial function we can view it also as a function on $\mathbb{R}^+$. For all $m \in \mathbb{N}$ and $\varepsilon > 0$, define the shifted function

$$u_\varepsilon^m(x) \triangleq \begin{cases} u_\varepsilon^*(x) - u_\varepsilon^*(1/m), & x \in B_{1/m} \setminus \{0\}, \\ 0, & x \in \Omega \setminus B_{1/m}; \end{cases}$$

then we have the following estimates.

Lemma 2.2: *There exist C_1, C_2 and $K > 0$, such that if $\varepsilon^{2(N-2)/(2-s)} m^{2\beta} < K$, then*

$$\|u_\varepsilon^m\|^2 \le A_s^{(N-s)/(2-s)} + C_1 \varepsilon^{2(N-2)/(2-s)} m^{2\beta}, \tag{2.1}$$

$$\int_\Omega \frac{|u_\varepsilon^m|^{2^*(s)}}{|x|^s} \ge A_s^{(N-s)/(2-s)} - C_2 \varepsilon^{2(N-s)/(2-s)} m^{2^*(s)\beta}. \tag{2.2}$$

Proof: Set

$$C_\varepsilon \triangleq \Big(\frac{2\varepsilon^2\beta^2(N-s)}{\sqrt{\bar\mu}}\Big)^{\sqrt{\bar\mu}/(2-s)}.$$

By the definition of $u_\varepsilon^m(x)$ we have

$$\int_\Omega |\nabla u_\varepsilon^m|^2 = \int_{\mathbb{R}^N} |\nabla u_\varepsilon^*|^2 - \int_{\mathbb{R}^N\setminus B_{1/m}} |\nabla u_\varepsilon^*|^2 \le \int_{\mathbb{R}^N} |\nabla u_\varepsilon^*|^2, \tag{2.3}$$

$$\begin{aligned}
&\int_\Omega \frac{|u_\varepsilon^m|^2}{|x|^2}\\
&= \int_{B_{1/m}} \frac{|u_\varepsilon^*|^2}{|x|^2} + \int_{B_{1/m}} \frac{C_\varepsilon^2}{(1/m)^{2\gamma'}[\varepsilon^2+(1/m)^{(2-s)\beta/\sqrt{\bar\mu}}]^{2(N-2)/(2-s)}}\frac{1}{|x|^2}\\
&\quad - 2\int_{B_{1/m}} \frac{C_\varepsilon u_\varepsilon^*}{(1/m)^{\gamma'}[\varepsilon^2+(1/m)^{(2-s)\beta/\sqrt{\bar\mu}}]^{(N-2)/(2-s)}}\frac{1}{|x|^2}\\
&\ge \int_{\mathbb{R}^N} \frac{|u_\varepsilon^*|^2}{|x|^2} - C\int_{1/m}^\infty \frac{\varepsilon^{4\sqrt{\bar\mu}/(2-s)}}{r^{2\gamma'}(\varepsilon^2+r^{(2-s)\beta/\sqrt{\bar\mu}})^{2(N-2)/(2-s)}}\frac{1}{r^2}r^{N-1}dr\\
&\quad - C\int_0^{1/m} \frac{\varepsilon^{4\sqrt{\bar\mu}/(2-s)}r^{N-3}dr}{r^{\gamma'}(\varepsilon^2+r^{(2-s)\beta/\sqrt{\bar\mu}})^{\frac{(N-2)}{(2-s)}}(1/m)^{\gamma'}[\varepsilon^2+(1/m)^{(2-s)\beta/\sqrt{\bar\mu}}]^{\frac{(N-2)}{(2-s)}}}.
\end{aligned}$$

From

$$\int_{1/m}^\infty \frac{\varepsilon^{4\sqrt{\bar\mu}/(2-s)}r^{N-3-2\gamma'}dr}{(\varepsilon^2+r^{(2-s)\beta/\sqrt{\bar\mu}})^{2(N-2)/(2-s)}} \le C\varepsilon^{4\sqrt{\bar\mu}/(2-s)}m^{2\beta}$$

and

$$\int_0^{1/m} \frac{\varepsilon^{4\sqrt{\bar\mu}/(2-s)}r^{N-3-\gamma'}dr}{(\varepsilon^2+r^{(2-s)\beta/\sqrt{\bar\mu}})^{(N-2)/(2-s)}(1/m)^{\gamma'}[\varepsilon^2+(1/m)^{(2-s)\beta/\sqrt{\bar\mu}}]^{(N-2)/(2-s)}} \le C\varepsilon^{4\sqrt{\bar\mu}/(2-s)}m^{2\beta},$$

we obtain

$$\int_\Omega \frac{|u_\varepsilon^m|^2}{|x|^2} \ge \int_{\mathbb{R}^N} \frac{|u_\varepsilon^*|^2}{|x|^2} - C\varepsilon^{4\sqrt{\bar\mu}/(2-s)}m^{2\beta}.$$

Combining (2.3) and (1.7) we have

$$\begin{aligned}
\|u_\varepsilon^m\|^2 &= \int_\Omega |\nabla u_\varepsilon^m|^2 - \mu\int_\Omega \frac{|u_\varepsilon^m|^2}{|x|^2}\\
&\le \int_{\mathbb{R}^N}\Big(|\nabla u_\varepsilon^*|^2 - \mu\frac{|u_\varepsilon^*|^2}{|x|^2}\Big) + C\varepsilon^{4\sqrt{\bar\mu}/(2-s)}m^{2\beta}\\
&= A_s^{(N-s)/(2-s)} + C\varepsilon^{4\sqrt{\bar\mu}/(2-s)}m^{2\beta},
\end{aligned}$$

and (2.1) follows. In order to prove (2.2), noting that

$$\begin{aligned}
&\int_\Omega \frac{|u_\varepsilon^m|^{2^*(s)}}{|x|^s} \\
&= \int_{B_{1/m}} \frac{|u_\varepsilon^m|^{2^*(s)}}{|x|^s} \\
&\geq \int_{B_{1/m}} \frac{|u_\varepsilon^*|^{2^*(s)}}{|x|^s} \\
&\quad - 2^*(s) \int_{B_{1/m}} \frac{|u_\varepsilon^*|^{2^*(s)-1}}{|x|^s} \frac{C_\varepsilon}{(1/m)^{\gamma'}[\varepsilon^2 + (1/m)^{(2-s)\beta/\sqrt{\bar{\mu}}}]^{(N-2)/(2-s)}} \\
&= \int_{\mathbb{R}^N} \frac{|u_\varepsilon^*|^{2^*(s)}}{|x|^s} - \int_{\mathbb{R}^N \setminus B_{1/m}} \frac{|u_\varepsilon^*|^{2^*(s)}}{|x|^s} \\
&\quad - 2^*(s) \int_{B_{1/m}} \frac{|u_\varepsilon^*|^{2^*(s)-1}}{|x|^s} \frac{C_\varepsilon}{(1/m)^{\gamma'}[\varepsilon^2 + (1/m)^{(2-s)\beta/\sqrt{\bar{\mu}}}]^{(N-2)/(2-s)}},
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{R}^N \setminus B_{1/m}} \frac{|u_\varepsilon^*|^{2^*(s)}}{|x|^s} &= \int_{1/m}^\infty \frac{C_\varepsilon^{2^*(s)} r^{N-1-s} dr}{r^{2^*(s)\gamma'} \left(\varepsilon^2 + r^{(2-s)\beta/\sqrt{\bar{\mu}}}\right)^{\frac{N-2}{2-s} 2^*(s)}} \\
&\leq C\varepsilon^{2(N-s)/(2-s)} m^{2^*(s)\beta}
\end{aligned}$$

and

$$\begin{aligned}
\int_{B_{1/m}} \frac{|u_\varepsilon^*|^{2^*(s)-1}}{|x|^s} &\frac{C_\varepsilon}{(1/m)^{\gamma'}[\varepsilon^2 + (1/m)^{(2-s)\beta/\sqrt{\bar{\mu}}}]^{(N-2)/(2-s)}} \\
&\leq C\varepsilon^{2(N-s)/(2-s)} m^{2^*(s)\beta},
\end{aligned}$$

by (1.7) we have

$$\begin{aligned}
\int_\Omega \frac{|u_\varepsilon^m|^{2^*(s)}}{|x|^s} &\geq \int_{\mathbb{R}^N} \frac{|u_\varepsilon^*|^{2^*(s)}}{|x|^s} - C\varepsilon^{2(N-s)/(2-s)} m^{2^*(s)\beta} \\
&= A_s^{(N-s)/(2-s)} - C\varepsilon^{2(N-s)/(2-s)} m^{2^*(s)\beta},
\end{aligned}$$

and (2.2) follows.

3. The variational characterization

The variational characterization is based on a linking argument. We recall that a sequence $\{u_m\} \subset H_0^1(\Omega)$ is called a PS sequence for J at level c if $J(u_m) \to c$ and $J'(u_m) \to 0$ in $H^{-1}(\Omega)$.

LEMMA 3.1: *Suppose $\{u_m\} \subset H_0^1(\Omega)$ is a PS sequence for J. Then there exists $u \in H_0^1(\Omega)$ such that $u_m \rightharpoonup u$ weakly, up to a subsequence, and $J'(u) = 0$. Moreover, if $J(u_m) \to c$ with $c \in (0, \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)})$, then u is a nontrival solution of* (1.5).

Proof: The proof is standard and we only sketch it. It's easy to show that $\{u_m\}$ is bounded in $H_0^1(\Omega)$ and there exists u such that $u_m \rightharpoonup u$, up to a subsequence. Furthermore, $J'(u) = 0$ by the weak continuity of J'.

Assume $c \in (0, \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)})$ and, by contradiction, $u \equiv 0$. As the term u_m^2 is subcritical, from $\langle J'(u_m), u_m\rangle = o(1)$ we get

$$\|u_m\|^2 - \int_\Omega \frac{|u_m|^{2^*(s)}}{|x|^s} = o(1). \tag{3.1}$$

By the definition of A_s we have

$$\|u_m\|^2 \geq A_s \Big(\int_\Omega \frac{|u_m|^{2^*(s)}}{|x|^s} \Big)^{2/2^*(s)}$$

and then

$$\|u_m\|^2 (1 - A_s^{-2^*(s)/2} \|u_m\|^{2^*(s)-2}) \leq o(1).$$

If $\|u_m\| \to 0$, we contradict $c > 0$. Therefore

$$\|u_m\|^2 \geq A_s^{(N-s)/(2-s)} + o(1).$$

By (3.1) we get

$$\begin{aligned} J(u_m) =& \frac{1}{2}\|u_m\|^2 - \frac{1}{2^*(s)} \int_\Omega \frac{|u_m|^{2^*(s)}}{|x|^s} + o(1) \\ =& \frac{2-s}{2(N-s)} \|u_m\|^2 + o(1) \\ \geq& \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)} + o(1), \end{aligned}$$

which contradicts

$$c < \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)}.$$

Thus $u \not\equiv 0$ and u is a nontrival solution of problem (1.5).

By Lemma 3.1, in order to prove Theorems 1.1 – 1.3, it suffices to build a PS sequence for J at a level strictly between 0 and $\frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)}$. We deal with the case where the functional J has a linking geometry.

LEMMA 3.2: *Assume that* $\lambda \in (\lambda_k, \lambda_{k+1})$ *for some* $\lambda_k, \lambda_{k+1} \in \sigma_\mu$, *let* $Q_m^\varepsilon \triangleq [(\bar{B}_R \cap H_m^-) \oplus [0, R]\{u_\varepsilon^m\}]$ *and* $\Gamma \triangleq \{h \in C(Q_m^\varepsilon, H_0^1(\Omega)); h(v) = v, \forall v \in \partial Q_m^\varepsilon\}$. *Then* J *admits a* PS *sequence at level*

$$c = \inf_{h\in\Gamma} \max_{v\in Q_m^\varepsilon} J(h(v)).$$

Proof: By the Sobolev–Hardy inequality (see [9]) and our equivalent norm in $H_0^1(\Omega)$, the proof is similar to that of Lemma 4 in [7]. The main components are to prove the following claims:

CLAIM 1: *There exist* $\alpha, \rho > 0$ *such that*

$$J(v) \geq \alpha, \quad \forall v \in \{u \in H^+; \|u\| = \rho\}.$$

CLAIM 2: *There exists* $R > \rho$, *such that* $\max_{v\in\partial Q_m^\varepsilon} J(v) \leq \omega_m$ *with* $\omega_m \to 0$ *as* $m \to \infty$.

By Claim 1 and Claim 2, J satisfies all the assumptions of the linking theorem except for the PS condition. Then by standard methods we obtain the desired results.

LEMMA 3.3: *Suppose* $\lambda_k, \lambda_{k+1} \in \sigma_\mu, \lambda_k < \lambda < \lambda_{k+1}, 0 \leq \mu \leq \bar{\mu} - 1$ *and* ε *small enough. We have*

$$\max_{t\geq 0} J(tu_\varepsilon^m) < \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)}.$$

Proof: By contradiction, assume that for any $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$J(t_\varepsilon u_\varepsilon^m) \geq \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)}. \tag{3.2}$$

Then we claim that $t_\varepsilon \to t_0 > 0$, up to a subsequence. Otherwise, assume that $t_\varepsilon \to +\infty$ up to a subsequence; then $J(t_\varepsilon u_\varepsilon^m) \to -\infty$ as $\varepsilon \to 0$, in contradiction with (3.2), thus $\{t_\varepsilon\}$ is bounded and there exists $t_0 \geq 0$ such that $t_\varepsilon \to t_0$ up to a subsequence. If $t_0 = 0$, by (2.1), (2.2) and the fact that $\lim_{\varepsilon\to 0} \int_\Omega |u_\varepsilon^m|^2 = 0$, we have

$$J(t_\varepsilon u_\varepsilon^m) = \frac{1}{2} t_\varepsilon^2 \|u_\varepsilon^m\|^2 - \frac{\lambda}{2} t_\varepsilon^2 \int_\Omega |u_\varepsilon^m|^2 - \frac{1}{2^*(s)} t_\varepsilon^{2^*(s)} \int_\Omega \frac{|u_\varepsilon^m|^{2^*(s)}}{|x|^s} = o(1),$$

which contradicts (3.2). So $t_\varepsilon \to t_0 > 0$ up to a subsequence if (3.2) holds. Setting

$$g(t) \triangleq \frac{1}{2} t^2 \|u_\varepsilon^m\|^2 - \frac{1}{2^*(s)} t^{2^*(s)} \int_\Omega \frac{|u_\varepsilon^m|^{2^*(s)}}{|x|^s}, \quad t \in [0, +\infty),$$

$g(t)$ attains its maximum at

$$t_1 \triangleq \|u_\varepsilon^m\|^{2/(2^*(s)-2)}\left(\int_\Omega \frac{|u_\varepsilon^m|^{2^*(s)}}{|x|^s}\right)^{-1/(2^*(s)-2)},$$

$$g(t_1) = \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\|u_\varepsilon^m\|^{2\cdot 2^*(s)/(2^*(s)-2)}\left(\int_\Omega \frac{|u_\varepsilon^m|^{2^*(s)}}{|x|^s}\right)^{-2/(2^*(s)-2)}$$
$$\leq \frac{2-s}{2(N-s)}A_s^{(N-s)/(2-s)} + C\varepsilon^{2(N-2)/(2-s)}$$

as $\varepsilon \to 0$. So we have

(3.3)
$$\frac{1}{2}\|t_\varepsilon u_\varepsilon^m\|^2 - \frac{1}{2^*(s)}\int_\Omega \frac{|t_\varepsilon u_\varepsilon^m|^{2^*(s)}}{|x|^s} \leq \frac{2-s}{2(N-s)}A_s^{(N-s)/(2-s)} + C\varepsilon^{2(N-2)/(2-s)}.$$

Next, we estimate the lower order term $\int_\Omega |t_\varepsilon u_\varepsilon^m|^2$ for $\lambda_k < \lambda < \lambda_{k+1}$ and $0 \leq \mu \leq \bar\mu - 1$. For $q = 2^{1/\gamma'}$, we may take ε small enough so that

$$\varepsilon^{2\sqrt{\bar\mu}/(2-s)\gamma} < 1/qm.$$

Hence there exists $C > 0$ such that

$$|x|^{\gamma'}(\varepsilon^2 + |x|^{(2-s)\beta/\sqrt{\bar\mu}})^{(N-2)/(2-s)} \leq C|x|^\gamma, \quad \forall |x| \geq \varepsilon^{\sqrt{\bar\mu}/\gamma}.$$

On the other hand,

$$u_\varepsilon^*(x) \geq u_\varepsilon^*(1/qm) > 2u_\varepsilon^*(1/m), \quad \forall x \in B_{1/qm}.$$

So we deduce

$$\int_\Omega |t_\varepsilon u_\varepsilon^m|^2 \geq C\int_{\varepsilon^{\sqrt{\bar\mu}/\gamma}}^{1/qm}\left(u_\varepsilon^*(r) - u_\varepsilon^*\left(\frac{1}{m}\right)\right)^2 r^{N-1}dr$$
$$\geq C\int_{\varepsilon^{\sqrt{\bar\mu}/\gamma}}^{1/qm}(u_\varepsilon^*(r))^2 r^{N-1}dr \geq C\varepsilon^{2(N-2)/(2-s)}\int_{\varepsilon^{\sqrt{\bar\mu}/\gamma}}^{1/qm} r^{1-2\beta}dr.$$

For $0 \leq \mu < \bar\mu - 1$ and $\beta = \sqrt{\bar\mu - \mu} > 1$, we have

$$\int_\Omega |t_\varepsilon u_\varepsilon^m|^2 \geq C\varepsilon^{2(N-2)/(2-s)}\varepsilon^{(N-2)(1-\beta)/\gamma} = C\varepsilon^{2(N-2)/(2-s)-(N-2)(\beta-1)/\gamma}.$$

For $\mu = \bar\mu - 1$ and $\beta = \sqrt{\bar\mu - \mu} = 1$, we get

$$\int_\Omega |t_\varepsilon u_\varepsilon^m|^2 \geq C\varepsilon^{2(N-2)/(2-s)}|\ln\varepsilon|.$$

Thus as $0 \leq \mu \leq \bar{\mu} - 1$, we obtain from (3.3)

$$J(t_\varepsilon u_\varepsilon^m) < \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)},$$

which contradicts (3.2) and we can complete the proof.

4. Proof of theorems

In this section, we give the proofs of Theorems 1.1 – 1.3.

Proof of Theorem 1.1: From Lemma 3.2, since the identity Id $\in \Gamma$, we have

$$\inf_{h \in \Gamma} \max_{v \in Q_m^\varepsilon} J(h(v)) \leq \max_{v \in Q_m^\varepsilon} J(v).$$

By Lemmas 3.1 and 3.2, Theorem 1.1 follows if we can prove that for some $\varepsilon > 0$ and $m \in \mathbb{N}$,

$$\sup_{v \in Q_m^\varepsilon} J(v) < \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)}. \tag{4.1}$$

To the contrary we assume that

$$\sup_{v \in Q_m^\varepsilon} J(v) \geq \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)}, \quad \forall m \in \mathbb{N}, \ \forall \varepsilon > 0. \tag{4.2}$$

As the set $\{v \in Q_m^\varepsilon; J(v) \geq 0\}$ is compact, the supremum in (4.1) is attained. Therefore, for all $\varepsilon > 0$ there exists $w_\varepsilon \in H_m^-$ and $t_\varepsilon \geq 0$ such that for $v_\varepsilon \triangleq w_\varepsilon + t_\varepsilon u_\varepsilon^m$ we have

$$J(v_\varepsilon) = \sup_{v \in Q_m^\varepsilon} J(v) \geq \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)},$$

that is

$$\frac{1}{2}\|v_\varepsilon\|^2 - \frac{\lambda}{2}|v_\varepsilon|_2^2 - \frac{1}{2^*(s)} \int_\Omega \frac{|v_\varepsilon|^{2^*(s)}}{|x|^s} \geq \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)}, \quad \forall \varepsilon > 0. \tag{4.3}$$

By Claim 2 in the proof of Lemma 3.2, we obtain that the sequences $\{t_\varepsilon\} \subset \mathbb{R}^+$ and $\{w_\varepsilon\} \subset H_m^-$ are bounded. Up to subsequences we may assume that

$$t_\varepsilon \to t_0 \geq 0, \quad w_\varepsilon \to w_0 \in H_m^-.$$

The convergence of $\{w_\varepsilon\}$ can be viewed in any norm topology since the space H_m^- is finite dimensional. As $w_\varepsilon \in H_m^-$, by Lemma 2.1.(i) and the fact that

$\lambda \in (\lambda_k, \lambda_{k+1})$ we have

$$J(w_\varepsilon) = \frac{1}{2}\|w_\varepsilon\|^2 - \frac{\lambda}{2}|w_\varepsilon|_2^2 - \frac{1}{2^*(s)}\int_\Omega \frac{|w_\varepsilon|^{2^*(s)}}{|x|^s}$$
$$\leq \frac{\lambda_k + o(1)}{2}|w_\varepsilon|_2^2 - \frac{\lambda}{2}|w_\varepsilon|_2^2 \leq 0$$

for m large enough (from now on we maintain m fixed). By using (4.3) and by arguing as in Lemma 3.3 we have $t_0 > 0$, up to a subsequence. By Lemma 3.3 we have, as $\varepsilon \to 0$,

$$J(v_\varepsilon) = J(w_\varepsilon) + J(t_\varepsilon u_\varepsilon^m) \leq J(t_\varepsilon u_\varepsilon^m) < \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)},$$

which contradicts (4.2) and thus (4.1) holds. By Lemma 3.1 and Lemma 3.2, we get that problem (1.5) has a nontrivial solution $u \in H_0^1(\Omega)$. Since $\lambda > \lambda_1$, u must change sign in Ω, which means that $-u$ is also a sign-changing solution of problem (1.5).

Proof of Theorem 1.2: Set $\lambda_+ \triangleq \min\{\lambda_j \in \sigma_\mu; \lambda < \lambda_j\}$ and assume that

$$\lambda_+ - \lambda < A_s \left(\int_\Omega |x|^{2s/(2^*(s)-2)} \right)^{-(2-s)/(N-s)}.$$

For any $j \in \mathbb{N}$, let $M(\lambda_j)$ be the eigenspace corresponding to λ_j, let $M^+ \triangleq \oplus_{\lambda_j \geq \lambda_+} M(\lambda_j)$ (closure in $H_0^1(\Omega)$) and $M^- \triangleq \oplus_{\lambda_j \leq \lambda_+} M(\lambda_j)$.

Claim 3: *We have*

$$\beta_\lambda \triangleq \sup_{u \in M^-} J(u) \leq \frac{2-s}{2(N-s)}(\lambda_+ - \lambda)^{(N-s)/(2-s)} \int_\Omega |x|^{2s/(2^*(s)-2)}$$
$$< \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)};$$

furthermore, there exist $\rho_\lambda > 0$ and $\delta_\lambda \in (0, \beta_\lambda)$ such that $J(u) \geq \delta_\lambda$ for any $u \in M^+$ with $\|u\| = \rho_\lambda$.

Indeed, for any $u \in M^-$ we have $\|u\|^2 \leq \lambda_+ |u|_2^2$ and by Hölder's inequality we get

$$\begin{aligned} J(u) =&\frac{1}{2}\|u\|^2 - \frac{\lambda}{2}|u|_2^2 - \frac{1}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} \\ \leq&\frac{1}{2}(\lambda_+ - \lambda)|u|_2^2 - \frac{1}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} \\ \leq&\frac{1}{2}(\lambda_+ - \lambda)\Big(\int_\Omega |x|^{2s/(2^*(s)-2)}\Big)^{\frac{2^*(s)-2}{2^*(s)}}\Big(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\Big)^{2/2^*(s)} \\ &- \frac{1}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}. \end{aligned}$$

Since

$$\begin{aligned} &\max_{\rho\geq 0}\Big[\frac{1}{2}(\lambda_+ - \lambda)\Big(\int_\Omega |x|^{2x/(2*(s)-2)}\Big)^{(2^*(s)-2)/2^*(s)}\rho^2 - \frac{1}{2^*(s)}\rho^{2^*(s)}\Big] \\ &= \frac{2-s}{2(N-s)}(\lambda_+ - \lambda)^{(N-s)/(2-s)}\Big(\int_\Omega |x|^{2s/(2^*(s)-2)}\Big) \\ &< \frac{2-s}{2(N-s)}A_s^{(N-s)/(2-s)}, \end{aligned}$$

we obtain

$$\begin{aligned} \beta_\lambda \leq& \frac{2-s}{2(N-s)}(\lambda_+ - \lambda)^{(N-s)/(2-s)}\Big(\int_\Omega |x|^{2s/(2*(s)-2)}\Big) \\ <& \frac{2-s}{2(N-s)}A_s^{(N-s)/(2-s)}. \end{aligned}$$

Let $u \in M^+$. Utilizing the inequalities

$$\lambda_+|u|_2^2 \leq \|u\|^2 \quad \text{and} \quad A_s\Big(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\Big)^{2/2^*(s)} \leq \|u\|^2,$$

we have

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}|u|_2^2 - \frac{1}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} \geq \frac{\lambda_+ - \lambda}{2\lambda_+}\|u\|^2 - \frac{\|u\|^{2^*(s)}}{2^*(s)A_s^{2^*(s)/2}}.$$

Since

$$\begin{aligned} &\max_{\rho\geq 0}\Big(\frac{\lambda_+ - \lambda}{2\lambda_+}\rho^2 - \frac{1}{2^*(s)A_s^{2^*(s)/2}}\rho^{2^*(s)}\Big) \\ &\qquad = \frac{2-s}{2(N-s)}\Big(\frac{\lambda_+ - \lambda}{\lambda_+}\Big)^{(N-s)/(2-s)}A_s^{(N-s)/(2-s)} \triangleq \delta_0, \end{aligned}$$

the maximum is attained at the point

$$\rho_0 \triangleq \left(\frac{\lambda_+ - \lambda}{\lambda_+} A_s^{\frac{2^*(s)}{2}}\right)^{1/(2^*(s)-2)}.$$

If we take $\rho_\lambda = \rho_0$ and $\delta_\lambda < \delta_0$, then we have $J(u) \geq \delta_\lambda$ for all $u \in M^+ \cap \partial B_{\rho_\lambda}$. Since $M^+ \cap M^- = M(\lambda_+)$, we have $M^+ \cap M^- \cap \partial B\rho_\lambda \neq \emptyset$ and any $u \in M^+ \cap M^- \cap \partial B\rho_\lambda$ satisfies $\delta_\lambda < J(u) \leq \sup_{u \in M^-} J(u) = \beta_\lambda$, which completes the proof of Claim 3.

By arguments similar to that of [7], we can get our desired results and complete the proof of Theorem 1.2.

Proof of Theorem 1.3: The proof of Theorem 1.3 follows the same lines as that of Theorem 1.1; however, some refinements of the estimates are required. To emphasize the dependence on m, we denote $v_\varepsilon^m, u_\varepsilon^m$ and w_ε^m instead of $v_\varepsilon, u_\varepsilon$ and w_ε. To prove (4.1), arguing by contradiction we assume that (4.2) holds, i.e., for all m large enough and all $\varepsilon > 0$, there exist $v_\varepsilon^m \in Q_m^\varepsilon$ and $t_\varepsilon \geq 0$ such that

$$\frac{1}{2}\|v_\varepsilon^m\|^2 - \frac{\lambda_1}{2}|v_\varepsilon^m|_2^2 - \frac{1}{2^*(s)}\int_\Omega \frac{|v_\varepsilon^m|^{2^*(s)}}{|x|^s} \geq \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)}; \tag{4.4}$$

then the sequences $\{t_\varepsilon\}$ and $\{w_\varepsilon^m\}$ again satisfy

$$t_\varepsilon \geq C > 0 \quad \text{and} \quad \|w_\varepsilon^m\| \leq C. \tag{4.5}$$

In order to deal only with one parameter, set $\varepsilon = m^{-(N+2)(2-s)\beta/(2(N-2))}$. Then as $m \to \infty$, (2.1) and (2.2) become

$$\|u_\varepsilon^m\|^2 \leq A_s^{(N-s)/(2-s)} + C_1 m^{-N\beta}, \tag{4.6}$$

$$\int_\Omega \frac{|u_\varepsilon^m|^{2^*(s)}}{|x|^s} \geq A_s^{(N-s)/(2-s)} - C_2 m^{-N(N-s)\beta/(N-2)}. \tag{4.7}$$

Note that $m^{-N(N-s)\beta/(N-2)} = o(m^{-N\beta})$ as $m \to \infty$. As in the proof of Lemma 3.3, there exist $C_3, C_4 > 0$ such that

$$\varepsilon^2 + |x|^{(2-s)\beta/\sqrt{\bar{\mu}}} \leq C_3 |x|^{(2-s)\beta/\sqrt{\bar{\mu}}}, \quad \forall |x| \geq C_4 \varepsilon^{2\sqrt{\bar{\mu}}/(2-s)\beta}.$$

Let $q = 2^{1/\gamma'}$; then we have $u_\varepsilon^*(x) \geq u_\varepsilon^*(1/qm) > 2u_\varepsilon^*(1/m), \forall x \in B_{1/(qm)}$. Furthermore,

(4.8)
$$\begin{aligned}\int_\Omega |t_\varepsilon u_\varepsilon^m|^2 &\geq C \int_{C_4 \varepsilon^{2\sqrt{\bar{\mu}}/(2-s)\beta}}^{1/qm} |u_\varepsilon^*(r)|^2 r^{N-1} dr \\ &\geq C \cdot C_\varepsilon^2 \int_{C_4 \varepsilon^{2\sqrt{\bar{\mu}}/(2-s)\beta}}^{1/qm} r^{1-2\beta} dr \geq C\varepsilon^{4\sqrt{\bar{\mu}}/(2-s)\beta} = Cm^{-(N+2)}.\end{aligned}$$

From now on we denote by v^m, u^m and w^m the functions $v_\varepsilon^m, u_\varepsilon^m$ and w_ε^m with the above choice of ε and with t_m the corresponding t_ε.

CLAIM 4: *For $0 \le \mu < \bar{\mu} - ((N+2)/N)^2$, m large enough, we have*

$$J(t_m u^m) \le \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)} - Cm^{-(N+2)}.$$

Indeed, by (4.6) and (4.7) we have

$$\begin{aligned}
J(t_m u^m) =& \frac{1}{2}\|t_m u^m\|^2 - \frac{\lambda_1}{2}|t_m u^m|_2^2 - \frac{1}{2^*(s)}\int_\Omega \frac{|t_m u^m|^{2^*(s)}}{|x|^s} \\
\le& \frac{1}{2} t_m^2 (A_s^{(N-s)/(2-s)} + Cm^{-N\beta}) - Cm^{-(N+2)} \\
&- \frac{1}{2^*(s)} t_m^{2^*(s)} (A_s^{(N-s)/(2-s)} - Cm^{-\frac{N(N-s)\beta}{N-2}}) \\
=& A_s^{(N-s)/(2-s)} \left(\frac{t_m^2}{2} - \frac{t_m^{2^*(s)}}{2^*(s)}\right) \\
&+ Cm^{-N\beta} - Cm^{-(N+2)} + Cm^{-N(N-s)\beta/(N-2)} \\
\le& \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)} - Cm^{-(N+2)}
\end{aligned}$$

for m large enough, where we have used the fact that

$$\max_{t\ge 0}\left(\frac{t^2}{2} - \frac{t^{2^*(s)}}{2^*(s)}\right) = \frac{2-s}{2(N-s)}$$

and

$$N+2 < N\beta < \frac{N(N-s)}{N-2}\beta, \quad \text{for } 0 \le \mu < \bar{\mu} - \left(\frac{N+2}{N}\right)^2, \quad 0 \le s < 2.$$

Thus Claim 4 holds.

On the other hand, by Lemma 2.1.(ii) and Hölder's inequality we have

$$\begin{aligned}
J(w^m) &= \frac{1}{2}\|w^m\|^2 - \frac{\lambda_1}{2}|w^m|_2^2 - \frac{1}{2^*(s)}\int_\Omega \frac{|w^m|^{2^*(s)}}{|x|^s} \\
&\le C_5 m^{-2\beta}|w^m|_2^2 - C_6|w^m|_2^{2^*(s)}
\end{aligned}$$

for some $C_5, C_6 > 0$. Then there exists $C_7 > 0$ such that

$$\max_{t\ge 0}(C_5 m^{-2\beta}t^2 - C_6 t^{2^*(s)}) = C_7 m^{-2(N-s)\beta/(2-s)}.$$

Hence

$$J(w^m) \le Cm^{-2(N-s)\beta/(2-s)}.$$

From Claim 4 and the fact that $|\operatorname{supp}(u^m) \cap \operatorname{supp}(w^m)| = 0$, we deduce

$$\begin{aligned} J(v^m) &= J(t_m u^m) + J(w^m) \\ &\le \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)} + Cm^{-2(N-s)\beta/(2-s)} - Cm^{-(N+2)}. \end{aligned}$$

By $0 \le \mu < \bar{\mu} - ((N+2)/N)^2$, we get

$$N + 2 < N\beta < \frac{2(N-s)}{2-s}\beta.$$

Therefore,

$$J(v^m) < \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)}$$

for m large enough. This contradicts (4.4) and the proof of Theorem 1.3 is completed.

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